# Product of n Groups

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**ABSTRACT:**

In this paper we check the permutable product of supersoluble subgroups. And the end, we obtain sufficient conditions for permutable products of finite supersoluble groups to be supersoluble.

**Keywords:**

Supersoluble, Product groups, Finite groups

## 1 INTRODUCTION AND STATEMENT OF RESULTS:

Two subgroups A and B of a group G are called permutable if every subgroup X of A is permutable with every subgroup Y of B, i.e.,XYis a subgroup of G. In this case, if G=AB we say that G is the permutable product of the subgroups A and B. Throughout the paper only finite groups are considered.

It is known that products of supersoluble groups are not supersoluble in general. Consequently the problem of finding sufficient conditions to assure that a product of two supersoluble groups is supersoluble has received considerable attention. In this paper we prove the following theorem:

**Main Theorem and Quistions:**

If G=A1…An is product of the n subgroups A1, …, and An. Then G is properties of same groups ?

Main results The following four lemmas are needed to prove Theorem 1.

**Lemma 1**[4, Theorem 2]. If G=AB is the mutually permutable product of the supersoluble subgroups A and B, then G is soluble.

**Lemma 2**([12]). Let G=AB be the mutually permutable product of the supersoluble subgroups A and B. Then, either G is supersoluble or NA < G and NB < G for every minimal normal subgroup N of G.

**Lemma 3** ([12]). Let G=AB be the mutually permutable product of the subgroups A and B and let N be any minimal normal subgroup of G. Then either N∩A=N∩B=1 or N=(N∩A)(N∩B).

**Lemma 4**[8, Lemma A.13.6]. Let G be a group, and N a minimal normal subgroup of G such that |N|=pn, where p is a prime and n>1. Denote C=CG(N) and assume that G/Cis supersoluble. Then, if Q/Cis a subgroup of G/C containing Op′(G/C), we have that Q is normal in G and N=∏ti=1Ni, where Ni are non-cyclic minimal normal subgroups of NQ for i=1,...,t.

**Proof of Main Theorem.**
**(Proof by Induction)**
Let G=AB be the mutually permutable product of the supersoluble subgroups A and B, with CoreG(A∩B)=1, and suppose that G has been chosen minimal such that its supersoluble residual GU is non-trivial. Let N be a minimal normal subgroup of G contained in GU. Note that N is an elementary abelian p-group for some prime p. Applying Lemma 2, we have that both NA and NB are proper subgroups of G. Moreover, using Lemma 3, we have that either N=(N∩A)(N∩B ) or N∩A=N∩B=1. Assume first that N=(N∩A)(N∩B).
**(i)** If N∩A=1, then N is cyclic. Assume that N∩A=1. It follows that N is contained in B. Let N0 be a non-trivial cyclic subgroup of N. Since AN0 is a subgroup of G, we have that N0 =AN0∩N is anormal subgroup of AN0. Hence every cyclic subgroup of N is normalised by A. Now let N1 be a minimal normal subgroup of B contained in N. Since B is supersoluble, it follows
That N1 is cyclic and thus normalised by A. Hence N1 is a normal subgroup of G. The minimality of N implies that N=N1 and consequently N is cyclic.
**(ii)** N∩A=1 and N∩B=1.On the contrary, assume that N∩A=1. From (i), we know that N is cyclic. Moreover, Nis contained in B. Hence AN∩B= (A∩B)N. Let L=CoreG(A∩B)N). Clearly, N is contained in Land L=L∩((A∩B)N)=(L∩A∩B)N. It is clear that G/L=(AL/L)(BL/L)is a mutually permutable product of AL/L and BL/L suchthat CoreG/L((AL/L)∩(BL/L))=1. By the minimality of G, it follows that G/L is supersoluble. On the other hand, since N is cyclic, we have that G/CG(N) is abelian. Hence G/CL(N) is supersoluble and GUCL(N)=C. Note that C=N×(C∩A∩B). Therefore C∩A∩B contains a Hall p′-subgroup of C. Since CoreG(A∩B)=1 and Op′(C) is a normal subgroup of G contained in C∩A∩B, we have that Op′(C)=1. Moreover, C′=(C∩A∩B)′ is a normal subgroup of G contained in A∩B. Consequently, C′=1 and C is an abelian p-group. In particular, GU is abelian and thus GU is complemented in G by a supersoluble normalizer D which is also a supersoluble projector of G, by [8, Theorems V.4.2 and V.5.18]. Since N is cyclic, we know that N is central with respect to the saturated formation of all supersoluble groups. By [8,Theorem V.3.2.e], Dcovers N and thus N is contained in D. It follows ND∩GU=1, a contradiction.

**(iii)** Either N=N∩A or N=N∩B. If we have N=N∩A=N∩B, then N is contained in A∩B, contradicting the factthat CoreG(A∩B)=1. We may assume without loss of generality that N∩A=N.

**(iv)** AN and BN are both supersoluble. Since N=(N∩A)(N∩B) and N=N∩A, it follows that N∩B is not contained in N∩A. Let n be any element of N∩B such that n/∈N∩A, and write N0 = 〈n〉. Note that AN0 is a subgroup of G, and AN0∩N=(N∩A)N0. Therefore N0(N∩A) is a normal subgroup of AN0, and consequently A normalizes (A∩N)N0. This yields that A/CA(N/N∩A) acts as a power automorphism group on N/N∩A. This means that AN is supersoluble. If N∩B=N, then BN=B is supersoluble. On the contrary, if N∩B=N, we can argue as above and we obtain that BN is supersoluble. Consequently, ACG(N)/CG(N) and BCG(N)/CG(N) are both abelian groups of exponent dividing p−1. But then G/CG(N)=(ACG(N)/CG(N))(BCG(N)/CG(N)) is a π-group for some set of primes π such that if q∈π, then q divides p−1.

**(v)** Let B0 be a Hall π-subgroup of B. Then AB0∩N= A∩N.
This follows just by observing that AB0∩Nis contained in each Hall π′-subgroup of AB0 and every Hall π′-subgroup of A is a Hall π′-subgroup of AB0. Note that |G/CG(N)| is a π-number and AB0 contains a Hall π-subgroup of G. Therefore G=(AB0)CG(N). But then A∩N is a normal subgroup of G. The minimality of G yields either A∩N= 1or A∩N= N. This contradicts our assumption 1=N∩A=N, and so we cannot have N=(A∩N)(B∩N). Thus, by Lemma 3, we may assume N∩A=N∩B=1. Let M= CoreG(AN∩BN). Then N∩M= N and G/M is supersoluble by the minimality of G. Again, we reach a contradiction after several steps.

**(vi)** M=N. Suppose that M=N. Since G/M is supersoluble, we know that N cannot be cyclic. Let us write C=CG(N), and consider the quotient group G/C. It is clear that G/C is supersoluble. Let Q/C=Op(G/C). Since Op(G/C)=1 and (G/C)′is nilpotent, it follows that Q/C is a normal Hall p′-subgroup of G/C. Let Bp′ be a Hall p′-subgroup of B. Since |N| divides |B:A∩B|, we have that (A∩B)Bp′ is a proper subgroup of B. Let T be a maximal subgroup of B containing (A∩B)Bp′. Then AT is a maximal subgroup of G and |G:AT|= p= |B:T|. If N is not contained in AT, we have G=(AT )N and AT∩N=1. Then |N|=p, a contradiction. Therefore N is contained in AT. In particular, the family S={X:X is a proper subgroup of B, (A∩B)Bp′X and NAX} is non-empty. Let R be an element of S of minimal order. Observe that AR has p-power index in G and thus ARC/C contains Op′(G/C). Regarding N as a AR-module over GF (p), we know, by Lemma 4, that N is a direct sum N=∏ti=1Ni, where Ni is an irreducible AR-module whose dimension is greater than 1, for all i∈{1,...,t}. Assume that (A∩B)Bp′=R. Then AR=ABp′ and thus N is contained in A, a contradiction. Therefore ABp′∩B=(A∩B)Bp′ is a proper subgroup of R. Let S be a maximal subgroup of R containing (A∩B)Bp′.

From the minimality of R, we know that N is not contained in AS. Consequently, there exists some i∈{1,...,t} such that Ni is not contained in AS, which is a maximal subgroup of AR. Hence AR=(AS)Ni. Since Ni is a minimal normal subgroup of AR, it follows that AS∩Ni= 1and |Ni|= |AR:AS|= |R:S|= p, a contradiction.

**(vii)** M is an elementary abelian p-group. Note that M=N(M∩A)=N(M∩B) and |M∩A|=|M∩B|=|M|/|N|. Moreover, A(M∩B)is a subgroup of G such that A(M∩B)∩M= (M∩A)(M∩B). Hence (M∩A)(M∩B) is also a subgroup of G. If M∩A= M∩B, then M∩A is a normal subgroup of G contained in A∩B. This implies that M∩A=1 and consequently M=N, a contradiction. It yields that M∩A=M∩B. Next we see that (M∩A)(M∩B) is a normal subgroup of G. Since (M∩A)(M∩B)= M∩A(M∩B), we have that A normalizes (M∩A)(M∩B). Similarly, B normalises

(M∩A)(M∩B) since (M∩A)(M∩B)= M∩B(M∩A). This implies normality of (M∩A)(M∩B) in G. Let X=(M∩A)(M∩B). Since we cannot have M∩A= M∩B, M∩A must be strictly contained in X. Thus X=X∩M=(X∩N)(M∩A) > M∩A gives us X∩N=1. But then X∩N=N, giving NX. Suppose that Q is a Hall p′-subgroup of M∩B. Then QA is a subgroup and so QA∩M=Q(M∩A) is also a subgroup which contains Q. Hence, as |M:M∩A|=pk for some k, we have that QM∩A∩B. Thus QB∩MM∩A∩B and similarly QA∩MM∩A∩B. Consequently, QM is contained in M∩A∩B. Since QM=Op(M), it follows that Op( M) is a normal subgroup of G contained in A∩B. Hence Op(M)=1, a contradiction, and consequently Q=1andMis a p-group. Hence N is contained in Z(M) and M=N×(M∩A)=N×(M∩B). Thus φ(M)=φ(M∩A)=φ(M∩B) is a normal subgroup of G contained in A∩B. This implies that φ(M)=1 and M is an elementary abelian p-group, as claimed.(viii) Final contradiction. We have from the previous steps that M∩A is not contained in M∩B and that M∩B is not contained in M∩A because otherwise, since |M∩A|=|M∩B|, it follows that M∩A=M∩B is a normal subgroup of G contained in A∩B. This would imply M∩A=M∩B=1, and M=(M∩A)N=N. This fact contradicts step (vi).

Let x be an element of M∩B such that x/∈M∩A. Then A〈x〉 is a subgroup of G, and so is M0=A〈x〉∩M=(A∩M)〈x〉. Therefore M0 is an A-invariant subgroup of G. In particular, since M=(M∩A)(M∩B), we have that every subgroup of M/M∩A is A-invariant; that is, A/CA(M/M∩A) acts as a group of power automorphisms on M/M∩A. It is clear that M/M∩A is A-isomorphic to N. Consequently, A/CA(N) acts as a group of power automorphisms on N. This implies that A normalises each subgroup of N. A nalogously, B normalises each subgroup of N. It follows that N is a cyclic group. We argue as in step (ii) above to reach a final contradiction. We have that G/M is supersoluble and M is abelian.

Therefore GUM and thus GU is abelian and complemented in G by a supersoluble normaliser, D say, by [8, Theorem V.5.18]. Since N is cyclic, we know that D covers N and thus NGU∩D=1, a contradiction. Proof of Theorem 2.
Let M=GU denote the supersoluble residual of G. Theorem 1yields that G/CoreG(A∩B) is supersoluble. Therefore M is contained in CoreG(A∩B). In particular, M is supersoluble. Let F(M) be the Fitting subgroup of M. Since A and Bare supersoluble, we have that [M,A]F(A)∩MF(M) and [M,B]F(B)∩MF(M). Consequently, [M,G] is contained in F(M). Note now that the chief factors of G between F(M) and Mare cyclic,and recall that G/M is supersoluble. Therefore we have that G/F (M) is supersoluble. This implies that M=F(M) and thus M is nilpotent. Consequently, G/F (G) is supersoluble. We now show that G/F (G) is metabelian. We prove first that A′ and B′ both centralise every chief factor of G. Let H/K be a chief factor of G. If H/K is cyclic, then as G′ centralizes H/K, so do A′ an dB′. Hence we may assume that H/K is a non-cyclic p-chief factor of G for some prime p. Note that we may assume that H is contained in M because G/M is supersoluble and H/K is non-cyclic. To simplify notation, we can consider K=1. Since F(G) centralizes H [8, Theorem A.13.8.b], G/CG(H ) is supersoluble. Let Ap′ be a Hall p′-subgroup of A. By Maschke's theorem [8, Theorem A.11.5],H is a completely reducible Ap′-module and HAp′ is supersoluble because H is contained in A. Therefore Ap′/CAp′(H ) is abelian of exponent dividing p−1. This implies that the primes involved in |A/CA(H )| can only be p or divisors of p−1.The same is true for |B/CB(H )|. This implies that if p divides |G/CG(H )|, then p is the largest prime dividing |G/CG(H )|. But since Op(G/CG(H ))=1 and G/CG(H ) is supersoluble, it follows that G/CG(H ) must be a p′-group. Consider H as A-module over GF (p). Since ACG(H )/CG(H ) is a p′-group, we have that H is a completely reducible A-module and every irreducible A-submodule of H is cyclic. Consequently A′ centralizes H, and the same is true for B′. Let now U/V be a chief factor of G. Then G/CG(U/V )is the product of the abelian subgroups ACG(U/V )/CG(U/V ) and BCG(U/V )/CG(U/V ). By Itô's theorem [9], we have that G/CG(U/V )is metabelian. Since F(G)is the intersection of the centralisers of all chief factors (again by [8, Theorem A.13.8.b]), we can conclude that G/F (G) is metabelian.3. Final remarks Finally, Theorem 1 enables us to give succinct proofs of earlier results on mutually permutable products.

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