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Research Article

Construction of Cohn Triples and Applications

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Abstract

Cohn triples of matrices and their links with the theory of free groups of rank were discovered in 1955 by Harvey Cohn. A lot of consequences were developed for the modular group SL (2, Z) and the free subgroup **F₂** In the present article, we deal with a new construction of such triples and resulting Diophantine equations.

Key words: cohn triples; fricke relations; fibonacci numbers

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Introduction

We deal in the present article with the Markoff spectrum M as defined by [1]. The minimum $m(f)$ of an indefinite binary quadratic form

$$f(x,y) = ax^2 + bxy + cy^2$$

with real coefficients and positive discriminant $\Delta(f) = b^2 - 4ac$ is

$$m(f) = \inf \left| f(x,y) \right|,$$

where the infimum is taken over all the pairs of integers x, y not both zero. The set of values $m(f)/\sqrt{\Delta(f)}$ is defined as being the *Markoff spectrum* M .

· With the *reduction* of the form f to f_j , we have the possibility of computing the values $m(f)$ with *doubly infinite sequences* $(\cdots, a_{-j}, \cdots, a_{-1}, a_0, a_1, \cdots, a_i, \cdots).$

We have

$$\xi_j = \left[a_j, a_{j+1}, ..., a_{2j}, ... \right] > 1,$$

$$-1 < \xi_j' = -(1/\eta_j) = -\left[0, a_{j-1}, a_{j-2}, ..., a_0, ... \right] < 0,$$

$$(\lambda_j) f_j(x,y) = \lambda_j (x-\xi_j y)(x-\xi_j' y),\ \ \xi_j = a_j + \frac{1}{\xi_{j+1}},\ \ \xi_j' = a_j + \frac{1}{\xi_{j+1}'},$$

$$\frac{2}{L_j} = \xi_j - \xi_j', \inf_{j\in \mathbb{Z}}\left(\frac{L_j}{2}\right) = m(f_j)/\sqrt{\Delta(f_j)} = C(f_j) = C(f) = m(f)/\sqrt{\Delta(f)}.$$

We go from ξ_{j+1} to ξ_j, and from ξ_{j+1}' to ξ_j' with 2×2 matrix with integer coefficients. Its determinant is often 1, but it will be possible to find another value -1 for the determinant of our matrices, and corresponding to a matrix of $GL(2,\mathbb{Z})$. More important, we will use the transpose of M_2:

$$M_2 = \begin{bmatrix} 3\gamma & -1 \\ 1 & 0 \end{bmatrix},\ \ M_1 = M_2^T \text{ transpose of } M_2,\ \ \gamma \in \mathbb{N}^* \text{ fixed parameter.}$$

Also $M_1, M_2 \in SL(2,\mathbb{Z})$ (modular group) when $\det M_2 = 1$. When multiplied, these matrices yield

$$(-M_2^{-1})(-M_1) = -\begin{bmatrix} 1 & 0 \\ 6\gamma & 1 \end{bmatrix},\ \ (-M_1)(-M_2^{-1}) = -\begin{bmatrix} 1 & -6\gamma \\ 0 & 1 \end{bmatrix}.$$

Transposing the second equality and adding it to the first one, we find a relation similar to the Heisenberg relation:

$$(M_2^{-1})(M_1) + (M_1^{-1})(M_2) = -2\times 1_2.$$

In what follows, we generalize the computations made by Cohn [6], [7], [8], in à set of articles trying to approach Markoff's forms through modular functions:

$$(M_1^{-1})(M_2^{-1})(M_1)(M_2) = \begin{bmatrix} 36\gamma^2+1 & 6\gamma \\ 6\gamma & 1 \end{bmatrix}. \qquad (1)$$

We see that the trace of the last commutator is not a multiple of 3. So we try to generalize the result quoted in the mémoire [9], saying that

Property 1. − For two matrices $A, B \in SL(2,\mathbb{Z})$, the following are equivalent:

1/ The couple (A,B) generates the free group $F_2 = [SL(2,\mathbb{Z}), SL(2,\mathbb{Z})]$ in $SL(2,\mathbb{Z})$,

2/ The triple $((tr(A)/3, (tr(B)/3, (tr(AB)/3))$ is a solution of the Markoff equation $x^2+y^2+z^2 = 3xyz$.

3/ We have $tr([A,B]) = tr(ABA^{-1}B^{-1}) = -2$.

Moreover, if (A', B') is another generating system for $\mathbb{F}_2$, the free group generated by (A,B), there exists one and only one $N \in GL(2,\mathbb{Z}) = \{M \mid 2\times 2$ integer matrix and $\det M = \pm 1\}$ up to a sign, verifying the conditions

$$A' = NAN^{-1},\ \ B' = NBN^{-1},$$

if and only if we have

$$((tr(A)/3), (tr(B)/3), (tr(AB)/3)) = ((tr(A')/3), (tr(B')/3), (tr(A'B')/3)).$$

Proof. See [17] (Chap. 6). Prop. 4.1 page 170 for $1 \Rightarrow 2$ Prop. 4.3 page 174 for $2 \Rightarrow 1.$ Also [9] (Chap. 6). Prop. 6.0.1 page 57 for $1 \Rightarrow 2,$ Prop. 6.0.2 page 57 for $2 \Rightarrow 1.$ The equivalence $2 \Leftrightarrow 3$ is a consequence of the formula of Fricke $(FR1)$ ([17] page 160). For the remaining part: ([17] Chap. 6. prop. 5.1 page 175). W

2 A new construction of the Cohn triples

2.1 Initial attempt to build a Cohn triple

We could choose, in order to have $ABC = 1_2$:

$$B = M_1^{-1}M_2^{-1} = \begin{bmatrix} 1 & -3\gamma \\ -3\gamma & 9\gamma^2+1 \end{bmatrix},\ \ A = M_1 = \begin{bmatrix} 3\gamma & 1 \\ -1 & 0 \end{bmatrix},$$

$$C^{-1} = M_2^{-1} = AB = \begin{bmatrix} 0 & 1 \\ -1 & 3\gamma \end{bmatrix},\ \ BA = \begin{bmatrix} 6\gamma & 1 \\ -18\gamma^2-1 & -3\gamma \end{bmatrix}.$$

But this gives for C and A the same trace, which is limited enough, and a trace of B not a multiple of 3. Also:

$$ABC = M_1 M_1^{-1} M_2^{-1} M_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = 1_2.$$

$$CBA = M_2 M_1^{-1} M_2^{-1} M_1 = \begin{bmatrix} 36\gamma^2 + 1 & 6\gamma \\ 6\gamma & 1 \end{bmatrix}.$$

$$(M_2)(M_1^{-1})(M_2^{-1})(M_1) = (M_1^{-1})(M_2^{-1})(M_1)(M_2).$$

2.2 Successful consequences of the definitive choice

We keep $C = M_2$ and put $AB = A^\bullet B^\bullet = C^{\bullet -1} = C^{-1}$, two matrices being built with $A^\bullet$ and $B^\bullet$, the matrices to be determined. Let us write, with $\varepsilon = \pm 1$ and $\theta \in \mathbb{Z}$, and compute, where the interesting cases seem to be $\theta \neq 0$:

$$A^\bullet B^\bullet C = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad CB^\bullet A^\bullet = \begin{bmatrix} \varepsilon & \theta \\ 0 & \varepsilon \end{bmatrix}.$$

We start with

$$B^\bullet A^\bullet = \begin{bmatrix} 3\gamma & -1 \\ 1 & 0 \end{bmatrix}^{-1} \begin{bmatrix} \varepsilon & \theta \\ 0 & \varepsilon \end{bmatrix} = \begin{bmatrix} 0 & \varepsilon \\ -\varepsilon & 3\gamma\varepsilon - \theta \end{bmatrix}.$$

$$A^\bullet B^\bullet = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 3\gamma & -1 \\ 1 & 0 \end{bmatrix}^{-1} = \begin{bmatrix} 0 & 1 \\ -1 & 3\gamma \end{bmatrix} = C^{-1} = M_2^{-1}.$$

As $tr(B^\bullet A^\bullet) = tr(A^\bullet B^\bullet)$, we have, because we suppose $\theta \neq 0$ and $\varepsilon = \pm 1$:

$$3\gamma\varepsilon - \theta = 3\gamma, \text{ hence } \varepsilon = -1 \text{ and } \theta = -6\gamma.$$

We can give new parameters defining $A^\bullet$, and new ones defining $B^\bullet$:

$$B^\bullet A^\bullet = \begin{bmatrix} t & u \\ v & w \end{bmatrix} \begin{bmatrix} k & l \\ m & n \end{bmatrix} = \begin{bmatrix} kt + mu & lt + nu \\ kv + mw & lv + nw \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 3\gamma \end{bmatrix} = M_1^{-1}.$$

$$A^\bullet B^\bullet = \begin{bmatrix} k & l \\ m & n \end{bmatrix} \begin{bmatrix} t & u \\ v & w \end{bmatrix} = \begin{bmatrix} kt + lv & ku + lw \\ mt + nv & mu + nw \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 3\gamma \end{bmatrix} = M_2^{-1}.$$

We write these two equations in dimension 4 and invert the matrices, after defining δ and verifying that

$$\delta = \det CA^\bullet = \det \begin{bmatrix} 3\gamma & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} k & l \\ m & n \end{bmatrix} = kn - lm = \det A^\bullet \in \{\pm 1\},$$

$$1 = \det C^{-1} = \det A^\bullet B^\bullet = \delta \det B^\bullet, \quad \det B^\bullet = \delta \in \{\pm 1\}.$$

$$(B^\bullet A^\bullet): \begin{bmatrix} kt + mu \\ kv + mw \\ lt + nu \\ lv + nw \end{bmatrix} = \begin{bmatrix} k & m & 0 & 0 \\ 0 & 0 & k & m \\ l & n & 0 & 0 \\ 0 & 0 & l & n \end{bmatrix} \begin{bmatrix} t \\ u \\ v \\ w \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ -1 \\ 3\gamma \end{bmatrix},$$

$$\delta \begin{bmatrix} t \\ u \\ v \\ w \end{bmatrix} = \begin{bmatrix} n & 0 & -m & 0 \\ -l & 0 & l & 0 \\ 0 & n & 0 & -m \\ 0 & -l & 0 & l \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ -1 \\ 3\gamma \end{bmatrix} = \begin{bmatrix} m \\ -l \\ n - 3m\gamma \\ 3k\gamma - l \end{bmatrix}.$$

$$(A^\bullet B^\bullet): \begin{bmatrix} kt+lv \\ mt+nv \\ ku+lw \\ mu+nw \end{bmatrix} = \begin{bmatrix} k & 0 & l & 0 \\ m & 0 & n & 0 \\ 0 & k & 0 & l \\ 0 & m & 0 & n \end{bmatrix} \begin{bmatrix} t \\ u \\ v \\ w \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 1 \\ 3\gamma \end{bmatrix},$$

$$\delta \begin{bmatrix} t \\ u \\ v \\ w \end{bmatrix} = \begin{bmatrix} n & -l & 0 & 0 \\ 0 & 0 & n & -l \\ -m & k & 0 & 0 \\ 0 & 0 & -m & k \end{bmatrix} \begin{bmatrix} 0 \\ -1 \\ 1 \\ 3\gamma \end{bmatrix} = \begin{bmatrix} l \\ n-3l\gamma \\ -k \\ 3k\gamma - m \end{bmatrix}.$$

Now we solve:

$$\begin{bmatrix} m \\ -k \\ n-3m\gamma \\ 3k\gamma - l \end{bmatrix} = \begin{bmatrix} l \\ n-3l\gamma \\ -k \\ 3k\gamma - m \end{bmatrix} = \delta \begin{bmatrix} t \\ u \\ v \\ w \end{bmatrix}.$$

This gives $m = l$ and $n = 3l\gamma - k$ for $A^\bullet$, so an expression of $A^\bullet$ with two new parameters is as follows: $k = \beta$ and $l = \alpha$.

$$A^\bullet = \begin{bmatrix} k & l \\ l & 3l\gamma - k \end{bmatrix} = \begin{bmatrix} \beta & \alpha \\ \alpha & 3\alpha\gamma - \beta \end{bmatrix}, \quad (2)$$

For $B^\bullet$, it is also easy to write it with the same two parameters:

$$B^\bullet = \begin{bmatrix} t & u \\ v & w \end{bmatrix} = \delta \begin{bmatrix} \alpha & -\beta \\ -\beta & 3\beta\gamma - \alpha \end{bmatrix} = \begin{bmatrix} 3\beta\gamma - \alpha & \beta \\ \beta & \alpha \end{bmatrix}^{-1}, \quad (3)$$

$$\delta = \begin{bmatrix} \alpha & -\beta \\ -\beta & 3\beta\gamma - \alpha \end{bmatrix}^{-1} \begin{bmatrix} 3\beta\gamma - \alpha & \beta \\ \beta & \alpha \end{bmatrix}^{-1} = -\frac{1}{\alpha^2 - 3\gamma\alpha\beta + \beta^2} \in \{\pm 1_2\}.$$

$$\det A^\bullet = (\beta(3\alpha\gamma - \beta) - \alpha(\alpha)) = \delta^2(\alpha(3\beta\gamma - \alpha) - \beta(\beta) = \det B^\bullet = \delta \in \{\pm 1_2\}. \quad (4)$$

- A simpler calculation is possible, here presented in order to confirm the previous one:

$$\begin{bmatrix} t & u \\ v & w \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 3\gamma \end{bmatrix} \begin{bmatrix} k & l \\ m & n \end{bmatrix}^{-1} = \delta \begin{bmatrix} m & -k \\ n-3m\gamma & -(l-3k\gamma) \end{bmatrix},$$

$$\begin{bmatrix} t & u \\ v & w \end{bmatrix} = \begin{bmatrix} k & l \\ m & n \end{bmatrix}^{-1} \begin{bmatrix} 0 & 1 \\ -1 & 3\gamma \end{bmatrix} = \delta \begin{bmatrix} l & n-3l\gamma \\ -k & -(m-3k\gamma) \end{bmatrix},$$

$$m = l = \alpha, n - 3m\gamma = -k = n - 3l\gamma = -\beta.$$

Hence, we obtain (2), (3), (5) more completely than (4), and with the expressions for $B^\bullet$ and $A^\bullet$:

$$\det A^\bullet = \det B^\bullet = \delta = (-\alpha^2 + 3\gamma\alpha\beta - \beta^2) \in \{\pm 1\}. \quad (5)$$

$$B^\bullet = \begin{bmatrix} t & u \\ v & w \end{bmatrix} = M_1^{-1} \begin{bmatrix} k & l \\ m & n \end{bmatrix}^{-1} = \begin{bmatrix} k & l \\ m & n \end{bmatrix}^{-1} M_2^{-1},$$

$$B^{\bullet-1} = M_2 A^{\bullet} = A^{\bullet} M_1, \quad A^{\bullet-1} = M_1 B^{\bullet} = B^{\bullet} M_2. \qquad (6)$$

We find also some new expressions which are easy to establish:

$$A^{\bullet}\delta B^{\bullet} = \begin{bmatrix} \beta & \alpha \\ \alpha & 3\alpha\gamma - \beta \end{bmatrix}\begin{bmatrix} \alpha & -\beta \\ -\beta & 3\gamma\beta - \alpha \end{bmatrix}$$

$$= \delta \begin{bmatrix} 0 & 1 \\ -1 & 3\gamma \end{bmatrix} = \delta C^{-1}, \text{ or } A^{\bullet}B^{\bullet}C = 1_2,$$

$$\text{with } M_2^{-1} = \begin{bmatrix} 3\gamma & -1 \\ 1 & 0 \end{bmatrix}^{-1} = \begin{bmatrix} 0 & 1 \\ -1 & 3\gamma \end{bmatrix},$$

$$M_2 = B^{\bullet-1} A^{\bullet-1} \in SL(2,\mathbb{Z}). \quad (7)$$

$$\delta B^{\bullet}A^{\bullet} = \begin{bmatrix} \alpha & -\beta \\ -\beta & 3\gamma\beta - \alpha \end{bmatrix}\begin{bmatrix} \beta & \alpha \\ \alpha & 3\alpha\gamma - \beta \end{bmatrix} = \delta \begin{bmatrix} 0 & -1 \\ 1 & 3\gamma \end{bmatrix} = \delta C^{-1},$$

then, defining K:

$$\begin{bmatrix} 3\gamma & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & -1 \\ 1 & 3\gamma \end{bmatrix} = \begin{bmatrix} -1 & -6\gamma \\ 0 & -1 \end{bmatrix} = K, \text{ or } CB^{\bullet}A^{\bullet} = K,$$

where:

$$\begin{bmatrix} 3\gamma & -1 \\ 1 & 0 \end{bmatrix} = M_2, \begin{bmatrix} 0 & -1 \\ 1 & 3\gamma \end{bmatrix} = \begin{bmatrix} 3\gamma & 1 \\ -1 & 0 \end{bmatrix}^{-1} = M_1^{-1}, \; K = M_2 M_1^{-1}.$$

$$M_1 = A^{\bullet-1} B^{\bullet-1} \in SL(2,\mathbb{Z}). \qquad (8)$$

$$A^{\bullet}B^{\bullet}C = 1_2, \; CB^{\bullet}A^{\bullet} = K, \; K = B^{\bullet-1}A^{\bullet-1}B^{\bullet}A^{\bullet} = [B^{\bullet-1}, A^{\bullet-1}]. \qquad (9)$$

- The asymmetric position of δ at the front of $B^{\bullet}$, not $A^{\bullet}$, provokes a question. Replacing $B^{\bullet}$ by $B^{\bullet-1}$ is the answer to the question. The condition for M_1 and M_2 to be in $SL(2,\mathbb{Z})$ does not imply the same property for $A^{\bullet}$ and $B^{\bullet}$. These matrices are in $GL(2,\mathbb{ZZ})\backslash SL(2,\mathbb{ZZ})$ when their determinant δ is -1, and they are in $SL(2,\mathbb{ZZ})$ when $\delta = 1$. We have two cases, owing to the fact that δ can have two values, ± 1. In the two cases we exhibit a non trivial example.

Example 1. $-$ With $\beta = 5, \; \alpha = 2, \; \gamma = 1,$ we obtain $\delta = 1$:

$$A^{\bullet} = \begin{bmatrix} a & b \\ b & -a + 3\gamma b \end{bmatrix} = \begin{bmatrix} 5 & 2 \\ 2 & 1 \end{bmatrix}, \; B^{\bullet} = \begin{bmatrix} b & -a \\ -a & 3\gamma a - b \end{bmatrix} = \begin{bmatrix} 2 & -5 \\ -5 & 13 \end{bmatrix},$$

$$\delta = \det A^{\bullet} = -(a^2 - 3\gamma ab + b^2) = \det B^{\bullet} = 1, \; A^{\bullet}, B^{\bullet} \in SL(2,\mathbb{Z}).$$

$$tr(A^{\bullet})^2 + (trA^{\bullet}B^{\bullet})^2 + (trB^{\bullet})^2 - tr(A^{\bullet})tr(A^{\bullet}B^{\bullet})tr(B^{\bullet})$$

$$= 6^2 + 3^2 + 15^2 - 6 \times 3 \times 15$$

$$= 3^2(\beta^2 + \alpha^2 + 1^2 - 3 \times \beta \times \alpha \times 1) = 0.$$

We obtain a formula linking together the matrices $A^{\bullet}$ and $B^{\bullet}$, situated in $SL(2,\mathbb{Z})$. With $C^{\bullet} = C = A^{\bullet}B^{\bullet}$, this gives the triple $(B^{\bullet-1}, A^{\bullet}, B^{\bullet-1}A^{\bullet-1})$ introduced in ([17] Chap. 6. page 162), associated to $(5,2,1)$. W

Example 2. $-$ With $\beta = 35, \; \alpha = 6, \; \gamma = 2,$ we obtain $\delta = -1$ and :

$$A^{\bullet}=\begin{bmatrix}\beta & \alpha\\ \alpha & -\beta+3\gamma\alpha\end{bmatrix}=\begin{bmatrix}35 & 6\\ 6 & 1\end{bmatrix},$$

$$B^{\bullet}=\delta\begin{bmatrix}\alpha & -\beta\\ -\beta & 3\gamma\beta-\alpha\end{bmatrix}=-\begin{bmatrix}6 & -35\\ -35 & 204\end{bmatrix},$$

$$tr(A^{\bullet})^2-(trA^{\bullet}B^{\bullet})^2+(trB^{\bullet})^2-tr(A^{\bullet})tr(A^{\bullet}B^{\bullet})tr(B^{\bullet})$$

$$=36^2-6^2+210^2-36\times 6\times 210=0$$

$$=6^2(\beta^2-3\times 2\times\beta\times\alpha+\alpha^2)-6^2.$$

We obtain a formula linking together the matrices $A^{\bullet}$ and $B^{\bullet}$, which are situated in $GL(2,\mathbb{Z})\backslash SL(2,\mathbb{Z})$. W
In both cases, we can evaluate δ :

Property 2. − For $A^{\bullet},B^{\bullet}\in GL(2,\mathbb{Z}),$ identified before, we have the relation of Fricke with signs:

$$tr(A^{\bullet})^2+\delta(trA^{\bullet}B^{\bullet})^2+(trB^{\bullet})^2-tr(A^{\bullet})tr(A^{\bullet}B^{\bullet})tr(B^{\bullet})=0. \qquad (10)$$

Proof. We compute:

$$tr(A^{\bullet})^2+\delta(trA^{\bullet}B^{\bullet})^2+(trB^{\bullet})^2-tr(A^{\bullet})tr(A^{\bullet}B^{\bullet})tr(B^{\bullet})$$

$$=(3\gamma\alpha)^2+\delta(-3\gamma\delta\left(\beta^2-3\gamma\beta\alpha+\alpha^2\right))^2+(3\gamma\beta)^2$$

$$-(3\gamma\alpha)(-3\gamma\delta\left(\beta^2-3\gamma\beta\alpha+\alpha^2\right)(3\gamma\beta))$$

$$=(3\gamma)^2((\alpha)^2+\delta+(\beta)^2-(\alpha)(\beta)(3\gamma))$$

$$=(3\gamma)^2((\alpha)^2-(\alpha)(\beta)(3\gamma)+(\beta)^2+\delta)=(3\gamma)^2(-\delta+\delta)=0. \quad \text{W}$$

This proves the claim.

3 Links with the Fibonacci numbers

Two cases have been defined owing to the fact that δ can have two values, ± 1. To determine α and β, we got a Diophantine equation (4) which is easy to solve.

3.1 First case ($\delta=1$)

We find $\det(A^{\bullet})=\det(B^{\bullet})=1$, and $A^{\bullet}$, $B^{\bullet}\in SL(2,\mathbb{Z}).$ We deal with the Diophantine equation

$$(\alpha^2-3\alpha\beta+\beta^2)=-1.$$

It has been already studied in [2], and we have:

Property 3. − The Diophantine equation $(\alpha^2-3\gamma\alpha\beta+\beta^2)=-1$ has solutions if and only if $\gamma=\pm 1$.

Proof. The references ([2], Theorem 6.3.1. p. 150) [21], [16]) give all that is needed about the solutions. W

- Any solution (α,β) of this equation corresponds by a 1 to 1 correspondence to $(-\alpha,\beta)$, a solution of the equation $(\alpha^2-3\alpha\beta+\beta^2)=-1$. We have only to look at our equation $(\alpha^2-3\alpha\beta+\beta^2)=-1$, to get all the solutions of the other. Moreover, the matrices $A^{\bullet}$ and $B^{\bullet}$ are in $SL(2,\mathbb{Z})$. It is interesting to realize that with $\gamma=1$,

$$trA^{\bullet}=3b,\ trB^{\bullet}=3a,\ \ trA^{\bullet}B^{\bullet}=3,\ \ tr(A^{\bullet}B^{\bullet}A^{\bullet-1}B^{\bullet-1})=-2.$$

Applying Fricke's formula ([17] p. 160, Prop. 2) and simplifying,

$$\alpha^2-3\alpha\beta+\beta^2=-1=-\delta. \qquad (11)$$

This equation is solvable in integers with a method obtained from the classical Markoff theory. The solutions are written with the Fibonacci sequence (OEIS **A000045**). We find in [2], [16], all the solutions: $(1,1),(-1,-1)$ and for all $n\geq 1$:

$$(-F_{2n-1},-F_{2n+1}),\ \ (-F_{2n+1},-F_{2n-1}),\ \ (F_{2n-1},F_{2n+1}),\ \ (F_{2n+1},F_{2n-1}).$$

For $n = 1$, we get

$$(-F_1, -F_3) = (-1, -2), \ \ (-F_3, -F_1) = (-2, -1),$$

$$(F_1, F_3) = (1, 2), \ \ (F_3, F_1) = (2, 1).$$

For all the couples of solutions, if (α, β) is one of them $(-\alpha, -\beta)$ is another:

$(\beta, \alpha), \ (3\alpha - \beta, \alpha), \ (\beta, 3\beta - \alpha)$ and $(-\beta, -\alpha), \ (-\beta, \alpha - 3\beta), \ (\beta - 3\alpha, -\alpha)$ solutions.

We find a figure with four sequences connected at the end of each other, at the singular solution $(1,1)$. But with **bisequences** $(F_n)_{n\in\mathbb{Z}}$, defined as indexed by $\mathbb{Z}$, the Fibonacci bisequence gives

$$\cdots, F_{-4} = -3, F_{-3} = 2, F_{-2} = -1, F_{-1} = 1, \ F_0 = 0, F_1 = 1, F_2 = 1, \cdots$$

$$\forall n \in \mathbb{ZZ}, \ F_{2n+1} = F_{-(2n+1)}, \ \ F_{-2n} = -F_{2n}.$$

Hence we can write, on the upper infinite dihedral group C_∞, a bisequence (F_{2n+1}, F_{2n-1}) to name the nodes of C_∞, but this constrains us to use the second group C_∞ for the other bisequence $(-F_{2n+1}, -F_{2n-1})$. We will give another notation in the sequel, where (F_{2n+1}, F_{2n-1}) will be replaced by (F_{2n-1}, F_{2n+1}) if and only n is even, and so on for the three couples obtained by permutation of F_{2n+1} and F_{2n-1}, and multiplication of the two terms of the couple by -1. With this method we find a bisequence of pairs of positive Fibonacci numbers which are the positive solutions of the equation $(\alpha^2 - 3\alpha\beta + \beta^2) = -1.$ This corresponds to the infinite cyclic group $C_{+\infty}$, in the upper position in Figure 1.

	$(F_5, F_7) = (5,13)$	$(F_7, F_5) = (13,5)$
Z		[
$(2,5) = (F_3, F_5)$		$(5,2) = (F_5, F_3)$
Z	$C_{+\infty}$	[
$(1,2) = (F_1, F_3)$		$(2,1) = (F_3, F_1)$
^		[
	$(1,1) = (F_1, F_1)$	
C_2 ↑↓	C_2 ↑↓ singular	C_2 ↑↓
	[$(-1,-1) = (F_{-1}, F_{-1})$ ^	
$(-1,-2) = (F_{-1}, F_{-3})$	$C_{-\infty}$	$(-2,-1) = (-F_{-3}, F_{-1})$

Figure: n°1: *Set of solutions of* $\alpha^2 - 3\alpha\beta + \beta^2 = -1$

We see with the negative Fibonacci numbers a structure of a group isomorphic to $C_{+\infty} \times C_2 \simeq \mathbb{ZZ} \times \mathbb{ZZ}/2\mathbb{ZZ}$. We can also say that the matrix M_2 operates on the set of solutions.

Remark 1. − We have given in [20] the relation

$$F_{6n-9} = 3F_{6n-7} - F_{6n-5}.$$

Together with the same relation for F_{6n-7}

$$\begin{bmatrix} 3 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} F_{6n-7} \\ F_{6n-5} \end{bmatrix} = \begin{bmatrix} F_{6n-9} \\ F_{6n-7} \end{bmatrix} \text{ where } \begin{bmatrix} 3 & -1 \\ 1 & 0 \end{bmatrix} = M_2,$$

it gives relations fixing the orientation of the action of M_2, realizing the infinite cyclic group C_∞:

$$\ldots, \begin{bmatrix} 3 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \ldots$$

The commutative group $C_2 \times C_{+\infty} \simeq \mathbb{ZZ}/2\mathbb{ZZ} \times \mathbb{ZZ}$, operates on solutions of the equation $\alpha^2 - 3\alpha\beta + \beta^2 = -1.$ It is not the corresponding infinite dihedral group $D_\infty \simeq C_2 \times C_{+\infty} \simeq \mathbb{ZZ}/2\mathbb{ZZ} \rtimes \mathbb{ZZ}$, not studied here. W

3.2 Second case ($\delta = -1$)

We find $\det(A^\bullet) = \det(B^\bullet) = -1$, and $A^\bullet,\ B^\bullet \in GL(2,\mathbb{Z})\setminus SL(2,\mathbb{Z})$. We deal with the Diophantine equation

$$(\alpha^2 - 3\gamma\alpha\beta + \beta^2) = 1 = -\delta. \qquad (12)$$

This equation has been studied in [2] (pp. 130–150), and we have:

Property 4. − The Diophantine equation $(\alpha^2 - 3\gamma\alpha\beta + \beta^2) = 1$ has solutions if and only if $|3\gamma| \geq 2$.

Proof. The reference [2] gives all that is needed about the solutions. W

- We deal with matrices situated in $GL(2,\mathbb{Z})$ but not in $SL(2,\mathbb{Z})$. We give the continued fraction of η, the root of

$\psi_\gamma(X,1) = X^2 - 3\gamma X + 1$:

$$\eta = \frac{3\gamma + \sqrt{(9\gamma^2 - 4)}}{2} = [3\gamma - 1, \underline{1, 3\gamma - 2}].$$

We produce then the classical table of the values of the associated form. For $\gamma = 1$ we have $2 - 3\gamma = -1$, and we find with the following table two classes of solutions, couples of Fibonacci numbers up to signs, of the equation $\alpha^2 - 3\gamma\alpha\beta + \beta^2 = -1$ of the **first case**:

$3\gamma - 1] = 3\gamma - 1 = \dfrac{p}{q}$	:	$p^2 - 3\gamma pq + q^2 = 2 - 3\gamma$
$3\gamma - 1, 1] = \dfrac{3\gamma}{1} = \dfrac{\alpha}{\beta}$	:	$\alpha^2 - 3\gamma\alpha\beta + \beta^2 = 1$
$3\gamma - 1, 1, 3\gamma - 2] = \dfrac{(9\gamma^2 - 3\gamma - 1)}{(3\gamma - 1)} = \dfrac{p}{q}$	:	$p^2 - 3\gamma pq + q^2 = 2 - 3\gamma$
$3\gamma - 1, 1, 3\gamma - 2, 1] = \dfrac{(9\gamma^2 - 1)}{(3\gamma)} = \dfrac{p}{q}$	:	$p^2 - 3\gamma pq + q^2 = 1$

We see that the matrix $C = M_2$ plays an important role for the transportation of the period of η :

$$\begin{bmatrix} 3\gamma - 1 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 3\gamma - 2 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 3\gamma - 1 & 1 \\ 1 & 0 \end{bmatrix}^{-1}$$

$$= \begin{bmatrix} 3\gamma & -1 \\ 1 & 0 \end{bmatrix} = C = M_2.$$

This gives all the solutions of the equation $\alpha^2 - 3\gamma\alpha\beta + \beta^2 = -1$, with a sign ± 1 corresponding to the cycle C_2 and the infinite cycle $C_{+\infty}$ given by M_2 :

$$\ldots,\ M_2\begin{bmatrix} 0 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix},\ M_2\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 3\gamma \\ 1 \end{bmatrix},\ M_2\begin{bmatrix} 3\gamma \\ 1 \end{bmatrix} = \begin{bmatrix} 9\gamma^2 - 1 \\ 3\gamma \end{bmatrix},\ \ldots$$

In the present case, $\det(A^\bullet) = \det(B^\bullet) = -1$. We would like to be able to apply some relation similar to Fricke's formula, for example, the last expression of [17] (p. 28).

3.3 A general Fricke's equation

But a formula such as that one which will be true for $GL(2,\mathbb{Z}\mathbb{Z})$ is more complicated, and does not seem to be given in any of the numerous articles written about Fricke's formula. Working on this, we found:

Property 5. − For any matrices $A, B \in GL(2,\mathbb{Z})$, we have the generalized Fricke's formula:

$$tr[A,B]+2 = tr(ABA^{-1}B^{-1})+2$$
$$= \det(A)\times tr(A)^2+\det(B)\times tr(B)^2+\det(A)\times\det(B)\times tr(AB)^2$$
$$-\det(A)\times\det(B)\times tr(A)\times tr(B)\times tr(AB).$$

Proof. Let

$$A=\begin{bmatrix}\beta & \alpha\\ m & n\end{bmatrix},\quad B=\begin{bmatrix}t & u\\ v & w\end{bmatrix},\quad AB=\begin{bmatrix}t\beta+v\alpha & u\beta+w\alpha\\ mt+nv & mu+nw\end{bmatrix},$$

$$trA=\beta+n,\quad \det A=\beta n-\alpha m,\quad trB=t+w,\quad \det B=tw-uv,$$

$$tr(AB)=t\beta+v\alpha+mu+nw,\quad \det AB=\det A\det B,$$

$$ABA^{-1}B^{-1}$$

$$=\begin{bmatrix}\beta t+\alpha v & \beta u+\alpha w\\ mt+nv & mu+nw\end{bmatrix}(\beta n-\alpha m)^2\begin{bmatrix}\beta & \alpha\\ m & n\end{bmatrix}^{-1}(tw-uv)^2\begin{bmatrix}t & u\\ v & w\end{bmatrix}^{-1}$$

$$=\begin{bmatrix}\beta t+\alpha v & \beta u+\alpha w\\ mt+nv & mu+nw\end{bmatrix}(\beta n-\alpha m)\begin{bmatrix}n & -\alpha\\ -m & \beta\end{bmatrix}(tw-uv)\begin{bmatrix}w & -u\\ -v & t\end{bmatrix}$$

$$=(\beta n-\alpha m)(tw-uv)\begin{bmatrix}\beta t+\alpha v & \beta u+\alpha w\\ mt+nv & mu+nw\end{bmatrix}\begin{bmatrix}\alpha v+nw & -\alpha t-nu\\ -\beta v-mw & \beta t+mu\end{bmatrix},$$

$$tr(ABA^{-1}B^{-1})$$

$$=(\beta n-\alpha m)(tw-uv)(-\beta^2uv+\beta\alpha tv-\beta\alpha vw+\beta mtu-\beta muw+2\beta ntw+\alpha^2v^2$$
$$-\alpha mt^2-\alpha mw^2-\alpha ntv+\alpha nvw+m^2u^2-mntu+mnuw-n^2uv)$$
$$=(\beta n-\alpha m)(tw-uv)\Theta,$$
$$\Theta=-\beta^2uv+\beta\alpha tv-\beta\alpha vw+\beta mtu-\beta muw+2\beta ntw+\alpha^2v^2$$
$$-\alpha mt^2-\alpha mw^2-\alpha ntv+\alpha nvw+m^2u^2-mntu+mnuw-n^2uv).$$

Then

$$\det A\times tr(A)^2+\det B\times tr(B)^2$$
$$+\det A\times\det B\times tr(AB)^2-\det A\times\det B\times tr(A)tr(B)tr(AB)$$
$$=(\beta n-\alpha m)(\beta+n)^2+(tw-uv)(t+w)^2$$
$$+(\beta n-\alpha m)(tw-uv)(\beta t+\alpha v+mu+nw)^2$$
$$-((\beta n-\alpha m)(tw-uv)(\beta+n)(t+w)(\beta t+\alpha v+mu+nw))$$
$$=(\beta n-\alpha m)(\beta+n)^2+(tw-uv)(t+w)^2$$
$$-\left(tw-uv\right)\left(\beta n-\alpha m\right)\left(\beta w-\alpha v-mu+nt\right)\left(\beta t+\alpha v+mu+nw\right)$$
$$=(\beta n-\alpha m)(\beta+n)^2+(tw-uv)(t+w)^2+\left(tw-uv\right)\left(\beta n-\alpha m\right)\Xi,$$

With

$$\Xi = (\beta t + \alpha v + mu + nw)(\alpha v - \beta w + mu - nt)$$
$$= -\beta^2 tw + \beta\alpha tv - \beta\alpha vw + \beta mtu - \beta muw - \beta nt^2 - \beta nw^2 + \alpha^2 v^2$$
$$+2\alpha muv - \alpha ntv + \alpha nvw + m^2u^2 - mntu + mnuw - n^2tw,$$

and so we find

$$\Xi - \Theta = -(\beta^2 + n^2)(tw - uv) - (\beta n - \alpha m)(t^2 + w^2) - 2\beta ntw + 2\alpha muv$$
$$-(\beta + n)^2(tw - uv) - (\beta n - \alpha m)(t + w)^2$$
$$-2\beta ntw + 2\alpha muv + 2\beta n(tw - uv) + 2tw(\beta n - \alpha m)$$
$$= -(\beta + n)^2(tw - uv) - (\beta n - \alpha m)(t + w)^2 + 2\left(tw - uv\right)\left(\beta n - \alpha m\right).$$

Now we combine:

$$-2 - tr(ABA^{-1}B^{-1}) + \det A \times \det B \times tr(AB)^2$$
$$-\det A \times \det B \times tr(A)tr(B)tr(AB)$$
$$= -2 + \left(tw - uv\right)\left(\beta n - \alpha m\right)(\Xi - \Theta)$$
$$= -2 + \left(tw - uv\right)\left(\beta n - \alpha m\right)(-(\beta + n)^2(tw - uv)$$
$$-(\beta n - \alpha m)(t + w)^2 + 2\left(tw - uv\right)\left(\beta n - \alpha m\right))$$
$$= -2 - ((\beta + n)^2(\beta n - \alpha m) + (tw - uv)(t + w)^2 + 2)$$
$$= -\det A \times tr(A)^2 - \det B \times tr(B)^2,$$

and we get

$$\det A \times tr(A)^2 + \det A \times \det B \times tr(AB)^2 + \det B \times tr(B)^2$$
$$-\det A \times \det B \times tr(A)tr(AB)tr(B) = 2 + tr(ABA^{-1}B^{-1}) \qquad \text{W}$$

Here, the commutator to deal with is $[A,B] = ABA^{-1}B^{-1}$. And we are in the **parabolic case** if and only if $tr([A,B]) = -2$.

Example 3. − With $\delta = 1$ and for example $\beta = 5, \ \alpha = 2, \ \gamma = 1$:

$$\delta = 1 = -(\beta^2 - 3\gamma\beta\alpha + \alpha^2) = det\, A^\bullet = det\, B^\bullet, \quad A^\bullet, B^\bullet \in SL(2, \mathbb{ZZ}).$$

$$tr(A^\bullet)^2 + (trA^\bullet B^\bullet)^2 + (trB^\bullet)^2 - tr(A^\bullet)tr(A^\bullet B^\bullet)tr(B^\bullet)$$
$$= 3^2(5^2 + 2^2 + 1^2 - 3 \times 5 \times 2 \times 1) = 0.$$

We are in the case of the positive Fricke's relation, linking together the matrices $A^\bullet$ and $B^\bullet$, situated in $SL(2,\mathbb{Z})$. With $C = A^\bullet B^\bullet$, the triple $(B^\bullet, A^\bullet B^\bullet, A^\bullet)$ introduced in ([17] Chap. 6. page 162), is associated to $(5,2,1)$. W

Example 4. − With $\delta = -1$ and for example $\beta = 35, \ \alpha = 6, \ \gamma = 2$:

$$\delta = -1 = -(\beta^2 - 3\gamma\beta\alpha + \alpha^2) = det\, A^\bullet = det\, B^\bullet \quad A^\bullet, B^\bullet \in GL(2, \mathbb{ZZ}) \backslash SL(2, \mathbb{ZZ}).$$

$$tr(A^\bullet)^2 - (trA^\bullet B^\bullet)^2 + (trB^\bullet)^2 - tr(A^\bullet)tr(A^\bullet B^\bullet)tr(B^\bullet)$$
$$= 6^2(35^2 - 3 \times 2 \times 35 \times 6 + 6^2) - 6^2 = 0.$$

We obtain a formula linking together the matrices $A^\bullet$ and $B^\bullet$, which are situated in $GL(2, \mathbb{ZZ}) \backslash SL(2, \mathbb{ZZ})$. W

Remark 2. The cases with which we deal in Property 2 and Property 5 are different. In the first case, $A^\bullet$ and $B^\bullet$ are linked with strong constraints by the common coefficients β and α, and their positions inside these matrices. On the contrary, Property 5 is true for any matrices $A, B \in GL(2,\mathbb{Z})$. W

Example 5. − With $\delta = 1$ and for example

$$A = \begin{bmatrix} 11 & 3 \\ 7 & 2 \end{bmatrix} \in SL(2,\mathbb{Z}), \quad B = \begin{bmatrix} 37 & 11 \\ 10 & 3 \end{bmatrix} \in SL(2,\mathbb{Z}),$$

$$tr([A,B]) = tr(ABA^{-1}B^{-1}) = tr\begin{bmatrix} -1298 & 4799 \\ -829 & 3065 \end{bmatrix} = 1767 \neq -2,$$

we are not in the parabolic case. Moreover, we verify Property 5:

$$\det A \times tr(A)^2 + \det A \times \det B \times tr(AB)^2 + \det B \times tr(B)^2$$

$$-\det A \times \det B \times tr(A)tr(AB)tr(B) = 2 + tr(ABA^{-1}B^{-1})$$

$$= 13^2 + 520^2 + 40^2 - 13 \times 520 \times 40 = 1769 = tr([A,B]) + 2. \quad \text{W}$$

4 Final result for the free group with two generators

We face the fact that the group $gp(A^\bullet, B^\bullet)$ generated by $A^\bullet$ and $B^\bullet$ is free. By Property 1 and $tr(A^\bullet B^\bullet A^{\bullet-1} B^{\bullet-1}) = -2$, $A^\bullet$ and $B^\bullet$ generate the free group $\mathbb{F}_2 = [SL(2,\mathbb{Z}), SL(2,\mathbb{Z})] = gp(A^\bullet, B^\bullet)$ in $SL(2,\mathbb{Z})$. This group countains $M_1 = A^{\bullet-1}B^{\bullet-1}$ and $M_2 = B^{\bullet-1}A^{\bullet-1}$.

Property 6. − The subgroup $gp(M_1, M_2)$ of $\mathbb{F}_2$ is free and isomorphic to $\mathbb{F}_2 = gp(A^\bullet, B^\bullet)$, but not equal to $\mathbb{F}_2$.

Proof. The group $gp(M_1, M_2)$ generated by M_1 and M_2 is a subgroup of $\mathbb{F}_2$, hence by the theorem of Nielsen–Schreier ([14] p. 92), it is a free subgroup of $\mathbb{F}_2$. But $tr[M_2, M_1] = 38$ confirms with Property 1 that (M_1, M_2) is not a system of generators of $\mathbb{F}_2$. W

- A confirmation that $gp(M_1, M_2)$ is a free group is **not** given by the properties of the commutator of M_1 and M_2:

$$tr(M_1)^2 + (trM_2)^2 + (trM_1M_2)^2 - tr(M_1)(trM_2)(trM_1M_2)$$

$$= 3^2 + 3^2 + 11^2 - 3 \times 3 \times 11 = 139 - 99 = 40.$$

and not through Property 1, because

$$[M_2, M_1] = M_2 M_1 M_2^{-1} M_1^{-1} =$$

$$= \begin{bmatrix} 3 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 3 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 3 & -1 \\ 1 & 0 \end{bmatrix}^{-1} \begin{bmatrix} 3 & 1 \\ -1 & 0 \end{bmatrix}^{-1}$$

$$= \begin{bmatrix} 19 & 60 \\ 6 & 19 \end{bmatrix}, \text{ or } tr[M_2, M_1] = 38.$$

The **rank**, which is the number of generators of a free group, is 2 for $\mathbb{F}_2$. The **index** of the subgroup $gp(M_1, M_2)$ of $\mathbb{F}_2$, denoted by $k = [\mathbb{F}_2 : gp(M_1, M_2)]$ may be used:

- <u>1/ Suppose *k* is infinite</u>. We are in a situation where $\mathbb{F}_2$ is a free group and $gp(M_1, M_2)$, not a group with one element, has infinite index in $\mathbb{F}_2$. Then $gp(M_1, M_2)$ is of infinite rank ([4] p. 355). But this is false, because this group has two generators M_1 and M_2 hence a rank less than 2. This case is impossible.

Note that in $\mathbb{F}_2$, the derived group,

$$D(\mathbb{F}_2) = gp([x,y] \mid x, y \in \mathbb{F}_2) \subset \mathbb{F}_2,$$

has infinite rank ([4] Théorème (9.39) p. 355 or [15] prop. 3.1. p. 13):

$$rank(D(\mathbb{F}_2)) = \infty.$$

- <u>2/ Suppose *k* is finite</u>. We have ([15] Proposition 3.9. p. 16):

$$k = [\mathbb{F}_2 : gp(M_1, M_2)] = \frac{rank(gp(M_1, M_2)) - 1}{rank(\mathbb{F}_2) - 1} = 1.$$

Because our free groups $gp(M_1, M_2)$ and $\mathbb{F}_2$ have two generators, the former relations give :

$$[\mathbb{F}_2 : gp(M_1, M_2)] = 1, \text{ then } \mathbb{F}_2 ; \quad gp(M_1, M_2).$$

$$rank(gp(M_1, M_2)) = [\mathbb{F}_2 : gp(M_1, M_2)] + 1 = 2.$$

The conclusion is that $\mathbb{F}_2 ; \quad gp(M_1, M_2)$, not $\mathbb{F}_2 = gp(M_1, M_2)$. It would be more comforting if $A^\bullet$ and $B^\bullet$ could be written with words in M_1 and M_2. The conclusion would be an equality. But this does not happen: only the isomorphism is sure. W

- Property 1 is verified with $A^\bullet$ and $B^\bullet$, not M_1 or M_2, and we have (6). If we could write $A^\bullet$ as a word of M_1 and M_2, $A^\bullet = A^\bullet(M_1, M_2)$, we could write $B^\bullet$ the same way, and conversely:

$$B^\bullet = M_1^{-1} A^{\bullet -1} = M_1^{-1} A^\bullet(M_1, M_2)^{-1} = B^\bullet(M_1, M_2) = A^{\bullet -1} M_2^{-1}, \quad (13)$$

$$A^\bullet = M_2^{-1} B^{\bullet -1} = M_2^{-1} B^\bullet(M_1, M_2)^{-1} = A^\bullet(M_1, M_2) = M_2^{-1} B^{\bullet -1}. \quad (14)$$

We would like to conclude that $A^\bullet \in gp(M_1, M_2)$ and $B^\bullet \in gp(M_1, M_2)$, so $\mathbb{F}_2 = gp(A^\bullet, B^\bullet) = gp(M_1, M_2)$. But this is not true, and we have only

$$A^\bullet \notin gp(M_1, M_2) \text{ and } B^\bullet \notin gp(M_1, M_2).$$

Remark

$$gp(A^\bullet, B^\bullet) = \mathbb{F}_2 ; \quad gp(M_1, M_2), \quad rank(\mathbb{F}_2) = 2,$$

$$[\mathbb{F}_2 : gp(M_1, M_2)] = k < \infty \ \Rightarrow \ rank(gp(M_1, M_2)) = k + 1 < \infty.$$

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