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Estimating the Gamma and the Q-Gamma Functions in the Neutrix Sense

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Abstract

In this paper, the concepts of neutrices and neutrix limit are used to estimate the values of gamma and incomplete gamma functions and their q-analogues when < q < 1 in the neutrix sense at zero and negative integers. Also the values of rth derivatives of these functions are estimate for r=1, 2, ••

Keywords: gamma function; incomplete gamma functions ; q-gamma function; incomplete q-gamma function; neutrices ; neutrix limit

1. Introduction

The incomplete q-gamma functions arise from Jackson q-integral for the q-gamma function [1]

$$\Gamma_q(\alpha)=\int_0^{\frac{1}{1-q}} t^{\alpha-1}E_q^{-tq}d_qt, \qquad \alpha>0,\ 0<q<1 \tag{1.1}$$

By decomposing it into a q-integral from 0 to x (lower incomplete q-gamma function), and another from x to $1/(1-q)$ (upper incomplete q-gamma function) [2]

$$\gamma_q(\alpha,x)=\int_0^{x} t^{\alpha-1}E_q^{-tq}d_qt, \qquad \alpha>0,\ x>0 \tag{1.2}$$

$$\Gamma_q(\alpha,x)=\int_x^{\frac{1}{1-q}} t^{\alpha-1}E_q^{-tq}d_qt, \qquad \alpha>0,\ x>0 \tag{1.3}$$

Gupta [3] defined a q-analogue of the incomplete gamma function such version in a slightly different form and he studied its important analytical properties and gave an application of it in statistical distribution theory.

When $q\to 1$ the definitions from (1.1) to (1.3) reduce to the definitions of the gamma and the incomplete gamma functions, respectively, as [4]

$$\Gamma(\alpha)=\int_0^{\infty} t^{\alpha-1}e^{-t}dt, \qquad \alpha>0 \tag{1.4}$$

$$\gamma(\alpha,x)=\int_0^{x} t^{\alpha-1}e^{-t}dt, \qquad \alpha>0,\ x>0 \tag{1.5}$$

$$\Gamma(\alpha,x)=\int_x^{\infty} t^{\alpha-1}e^{-t}dt, \qquad \alpha>0,\ x>0 \tag{1.6}$$

Salem [5] proved that $\gamma_q(\alpha,x)$ is an entire function for fixed complex variable x and for all complex $\alpha = 0,-1,-2,\cdots$ on the open unit disc $|q|<1$ and $\Gamma_q(\alpha,x)$ can also be continued analytically for all complex numbers $\alpha,x;|\arg(x)|<\pi$ by means of the expansions

$$\Gamma_q(0,x)=E_1(x,q)=\frac{1-q}{\ln q}\ln x-\gamma_q+\sum_{n=1}^{\infty}\frac{q^n\gamma_q(n,x)}{[n]_q!} \tag{1.7}$$

$$=\frac{1-q}{\ln q}\ln x-\gamma_q+\sum_{n=1}^{\infty}\frac{(-1)^{n-1}q^{\frac{n(n+1)}{2}}x^n}{[n]_q\cdot[n]_q!} \tag{1.8}$$

Where $E_1(x,q)$ is the q-analogue of the exponential integral and $\gamma_q=((1-q)/\ln q)\Gamma_q'(1)$ denotes the q-analogue of the Euler-Mascheroni constant. The author in [6] used this result to derive an important recursive formula for the complementary of incomplete q-gamma function $\Gamma_q(\alpha,x)$ at the negative integers as follows

$$\Gamma_q(-m,x)=\frac{(-1)^m q^{\frac{m(m+1)}{2}}}{[m]_q!}E_1(x,q)-E_q^{-x}\sum_{k=1}^{m}\frac{(1-q)^k}{(q^{-m},q)_k}x^{k-m-1} \tag{1.9}$$

Where m is a positive integer and $|\arg(x)|<\pi-\epsilon,\ 0<\epsilon<\pi$.

Here, the Jackson q-integral is defined in a generic interval $[a,b]$ as [7]

$$\int_a^b f(x)d_qx=\int_0^b f(x)d_qx-\int_0^a f(x)d_qx$$

Where

$$\int_0^a f(x)d_qx=a(1-q)\sum_{n=0}^{\infty}q^nf(aq^n) \tag{1.10}$$

Provided the sum converges absolutely and E_q^x is one of q-analogues of the exponential function defined as

$$E_q^x = \sum_{n=0}^{\infty} \frac{q^{\frac{n(n-1)}{2}} x^n}{[n]_q!} = \prod_{n=0}^{\infty} (1 + x(1-q)q^n) \tag{1.11}$$

Where $[a]_q = (1-q^x)/(1-q)$ refers to the basic number and $[m]_q! = [m]_q[m-1]_q \cdots [1]_q$ with $[0]_q! = 1$ is q-shefted factorial for positive integer m.

Divergent integral is an integral that has infinite limits of integration or an unbounded integrand and is either infinite or lacks a definite finite value. There are many methods used to handle such integral (for example: Hadamard finite part and the Cauchy principal value). Fisher developed the method, which is very similar to Hadamard method used to define the pseudo-functions, regarded as a particular application of the so called neutrix calculus developed by Van der Corput [8]. His method is based on discarding of unwanted infinite quantities from asymptotic expansions and has been widely exploited in connection with the problem of distributional multiplication and convolution. Fisher used the neutrices and neutrix limit to define gamma, beta and incomplete gamma functions [9-12]. Ozcag. et al [13,14] applied the neutrix limit to extend the definition of the incomplete beta function and its partial derivatives for negative integers. Salem [15,16] applied the neutrix and the neutrix limit to extend the definitions of q-gamma function (1.1) and incomplete q-gamma function (1.2) and their derivatives to negative integer values.

The main purpose of this paper is estimating the gamma and incomplete gamma functions and their q-analogues in the neutrix sense. Throughout this paper, we restrict the values of q, α and x to be real and $0 < q < 1$, $x > 0$. The neutrix and the neutrix limit are defined as

Neutrix. A neutrix N is defined as a commutative additive group of functions $f(\xi)$ defined on a domain N' with values in an additive group N'', where further if for some f in N, $f(\xi) = \gamma$ for all $\xi \in N'$, then $\gamma = 0$. The functions in N are called negligible functions.

Neutrix limit. Let N be a set contained in a topological space with a limit point *which* does not belong to N. If $f(\xi)$ is a function defined on N' with values in N'' and it is possible to find a constant c such that $f(\xi) - c \in N$, then c is called the neutrix limit of f as ξ tends to a and we write $N-\lim_{\xi \to a} f(\xi) = c$.

From now on, let N be the neutrix having domain $N' = \{\epsilon: 0 < \epsilon < \infty\}$ and range N'' the real numbers, with the negligible functions being finite linear sums of the functions

$$\epsilon^{\lambda} \ln^{r-1} \epsilon, \qquad \ln^r \epsilon \qquad (\lambda < 0,\ r = 1, 2,).$$

And all functions $o(\epsilon)$ which converge to zero in the normal sense as ϵ tends to zero [8].

2. A brief review of the gamma and q-gamma functions

It has been shown in [12] that the rth derivative of the gamma function (1.4) is defined by the neutrix limit for all real number α

$$\Gamma^{(r)}(\alpha) = N - \lim_{\epsilon \to 0} \int_{\epsilon}^{\infty} t^{\alpha-1} e^{-t} dt, \qquad r = 0, 1, 2, \cdots \tag{2.1}$$

It has been also shown in [15] that the rth derivative of the q-gamma function (1.1) is defined by the neutrix limit for all real value of α

$$\Gamma_q^{(r)}(\alpha) = N - \lim_{\epsilon \to 0} \int_{\epsilon}^{\frac{1}{1-q}} t^{\alpha-1} \ln^r t E_q^{-qt} d_q t, \qquad r = 0, 1, 2, ... \tag{2.2}$$

If we want to make the details of this definition more precise, we redefine as follows:

$$\Gamma_q^{(r)}(0) = N - \lim_{\epsilon \to 0} \int_{\epsilon}^{\frac{1}{1-q}} t^{-1} \ln^r t E_q^{-qt} d_q t, \tag{2.3}$$

for $r = 0,1,2,...; 0 < q \le 1$ and for $m = 1,2,...$

$$\Gamma_q^{(r)}(-m) = N - \lim_{\epsilon \to 0} \int_{\epsilon}^{\frac{1}{1-q}} t^{-m-1} \ln^r t E_q^{-qt} d_q t. \tag{2.4}$$

The rth partial derivatives with respect to α of the incomplete gamma function (1.5) and its q-analogue (1.2) are proved and defined in [9, 10] and [16], respectively, by the neutrix limit for all α and $x > 0$

$$\gamma^{(r)}(\alpha, x) = N - \lim_{\epsilon \to 0} \int_{\epsilon}^{x} t^{\alpha-1} \ln^r t e^{-t} dt, \qquad r = 0, 1, 2, \cdots. \tag{2.5}$$

$$\gamma_q^{(r)}(\alpha, x) = N - \lim_{\epsilon \to 0} \int_{\epsilon}^{x} t^{\alpha-1} \ln^r t E_q^{-qt} d_q t, \qquad r = 0, 1, 2, \cdots. \tag{2.6}$$

In particular, it has been proved in [16] that

$$\gamma_q(0, x) = \int_0^x t^{-1} \left[E_q^{-qt} - 1\right] d_q t - \frac{(1-q)\ln x}{\ln q}, \tag{2.7}$$

An interesting formula for incomplete q-gamma function has been derived in [16] as the recursive formula

$$\gamma_q(-m, x) + \frac{q^m}{[m]_q} \gamma_q(-m+1, xq) = \frac{(-1)^m q^{\frac{m(m+1)}{2}}}{[m]_q [m]_q!} - \frac{q^m}{[m]_q} x^{-m} E_q^{-xq}, \tag{2.8}$$

For $x > 0$ and $m = 1, 2, \cdots$.

3. The main results

The present section contains the main results in this paper which deals with the estimating the gamma and incomplete gamma functions and their q-analogues in the neutrix sense. The neutrix limit is used to derive the following results:

Theorem 3.1. The incomplete q-gamma function has the equation

$$\gamma_q(0,x) = -E_q(x,q) - \gamma_q, \qquad x > 0 \tag{3.1}$$

and the q-gamma function has also the equation

$$\Gamma_q(0) = -\gamma_q \tag{3.2}$$

Proof. The formula (2.7) yields for $x > 0$

$$\begin{aligned}\gamma_q(0,x) &= \int_0^x t^{-1}\left[E_q^{-qt}-1\right]d_qt - \frac{(1-q)\ln x}{\ln q}\\ &= \sum_{n=1}^{\infty}\frac{(-1)^n q^{\frac{n(n+1)}{2}}}{[n]_q!}\int_0^x t^{n-1}d_qt - \frac{(1-q)\ln x}{\ln q}\\ &= \sum_{n=1}^{\infty}\frac{(-1)^n q^{\frac{n(n+1)}{2}}x^n}{[n]_q[n]_q!} - \frac{(1-q)\ln x}{\ln q}\end{aligned}$$

The q-integral and summation are permuted due to their convergence. Inserting the expansion of q-exponential integral (1.8) into the previous equation to obtain (3.1) which equivalently $\Gamma_q(0) = -\gamma_q$.

Remark 3.2. From the equation (3.1) when letting $q \to 1$, we get the incomplete gamma function $\gamma(0, x) = E_1(x) - \gamma$ where $E_1(x)$ is the exponential integral defined as[4]

$$E_1(x) = \int_x^{\infty} t^{-1}e^{-t}dt, \qquad |\arg(x)| < \pi$$

And $\gamma = 0.57721\cdots$ is the Euler constant. Also the equation (3.2) tends to the same result obtained by [12] for gamma function as $q \to 1$, $\Gamma(0) = -\gamma$.

Theorem 3.3. The incomplete q-gamma function has the equation

$$\begin{aligned}\gamma_q(-m,x) &= \frac{(-1)^m q^{\frac{m(m+1)}{2}}}{[m]_q!}\left(H_{m,q} - E_1(x,q) - \gamma_q\right)\\ &\quad - \frac{(-1)^m q^{\frac{m(m+1)}{2}}}{[m]_q!}E_q^{-x}\sum_{i=1}^{m}(-1)^i q^{-\frac{i(i-1)}{2}}[i-1]_q!x^{-i}, \quad m = 1,2,\cdots,\end{aligned} \tag{3.3}$$

And the q-gamma function has the equation

$$\Gamma_q(-m) = \frac{(-1)^m q^{\frac{m(m+1)}{2}}}{[m]_q!}\left(H_{m,q} - \gamma_q\right), \quad m = 1,2,\cdots \tag{3.4}$$

Where $H_{m,q}$ is the q-analogue of the Harmonic number defined as

$$H_{m,q} = \sum_{i=1}^{m}\frac{1}{[i]}. \tag{3.5}$$

Proof. It is easy from (1.10) to prove that

$$\int_0^{xq^n} f(t)d_qt = \int_0^x f(t)d_qt - x(1-q)\sum_{i=0}^{n-1}q^i f(xq^i), \quad n = 1,2,\cdots$$

This leads to

$$\gamma_q^{(r)}(\alpha, xq^n) = \gamma_q^{(r)}(\alpha, x) - x^{\alpha}(1-q)\sum_{i=0}^{n-1} q^{i\alpha} \ln^r(xq^i) E_q^{-xq^{i+1}}, \quad r = 0, 1, 2, \cdots. \quad (3.6)$$

Therefore (2.8) can be given in the form

$$\gamma_q(-m, x) = \frac{-q^m}{[m]_q}\gamma_q(-m+1, x) + f(m) \quad (3.7)$$

Where

$$f(m) = \frac{(-1)^m q^{\frac{m(m+1)}{2}}}{[m]_q \cdot [m]_q!} + \frac{x^{1-m}(1-q)q^m}{[m]_q} E_q^{-xq} - \frac{x^{-m}q^m}{[m]_q} E_q^{-xq}$$
$$= \frac{(-1)^m q^{\frac{m(m+1)}{2}}}{[m]_q \cdot [m]_q!} - \frac{x^{-m}q^m}{[m]_q} E_q^{-x}. \quad (3.8)$$

By means of the iterative method, the equation (3.7) can be written in the form

$$\gamma_q(-m, x) = \frac{(-1)^m q^{\frac{m(m+1)}{2}}}{[m]_q!}\gamma_q(0, x) + \sum_{i=1}^{m} \frac{(-1)^{m-i} q^{\frac{(m-i)(m+i+1)}{2}}}{\prod_{r=1}^{m-i}[i+r]_q} f(i) \quad .(3.9)$$

A short calculation with using the well-known identity

$$(aq^i, q)_{m-i} = \frac{(a,q)_m}{(a,q)_i}, \qquad i = 0, 1, 2, \cdots, m$$

Would yield

$$\prod_{r=1}^{m-i}[i+r]_q = \frac{(q^{i+1}, q)_{m-i}}{(1-q)^{m-i}} = \frac{[m]_q!}{[i]_q!}.$$

Inserting the equation (3.1) and the previous result into (3.9) to obtain

$$\gamma_q(-m, x) = \frac{(-1)^m q^{\frac{m(m+1)}{2}}}{[m]_q!}(-E_q(x,q) - \gamma_q) + \frac{(-1)^m q^{\frac{m(m+1)}{2}}}{[m]_q!}\sum_{i=1}^{m}(-1)^i q^{-\frac{i(i+1)}{2}}[i]_q! f(i)$$

From (3.8) and the above equation, we obtain (3.3). The equation (3.4) comes immediately by putting $x = 1/(1-q)$ in (3.3) with noting that $E_q^{\frac{-1}{1-q}} = 0$. This ends the proof.

As a consequence of the previous theorem when letting $q \to 1$, we have

Corollary 3.4. The incomplete gamma function has the equation

$$\gamma(-m, x) = \frac{(-1)^m}{m!}(H_m - E_1(x) - \gamma) - \frac{(-1)^m e^{-x}}{m!}\sum_{i=1}^{m}(-1)^i (i-1)! x^{-i}, \qquad (3.10)$$

And the gamma function has the equation

$$\Gamma(-m) = \frac{(-1)^m}{m!}(H_m - \gamma). \quad (3.11)$$

Where $m = 1, 2, \cdots$ and H_m is the Harmonic number defined as

$$H_m = \sum_{i=1}^{m}\frac{1}{i}.$$

Theorem 3.5. For $r = 1, 2, \cdots$, the rth derivative of the incomplete q-gamma function can be estimated by the equation

$$\gamma_q^{(r)}(0,x) = \frac{q-1}{(r+1)\ln q}\left(\ln^{r+1} x E_q^{-x} + \gamma_q^{(r+1)}(1,x)\right) - \frac{\ln^r q}{r+1}\sum_{i=0}^{r-1}\binom{r+1}{i}\ln^{-i} q\gamma_q^{(i)}(0,x) \quad (3.12)$$

,

and for the q-gamma function by the equation

$$\Gamma_q^{(r)}(0) = \frac{q-1}{(r+1)\ln q}\Gamma_q^{(r+1)}(1) - \frac{\ln^r q}{r+1}\sum_{i=0}^{r-1}\binom{r+1}{i}\ln^{-i} q\Gamma_q^{(i)}(0). \quad (3.13)$$

Proof. The q-derivative $D_q f.(t)$ of a function f is given as

$$D_q f(t) = \frac{f(t) - f(tq)}{(1-q)t}, \quad q \neq 1, \; t \neq 0$$

Therefore, by using the binomial theorem, we can deduce for $r = 1, 2, \cdots$ that

$$D_q \ln^{r+1} t = \frac{\ln^{r+1} t - (\ln t + \ln q)^{r+1}}{(1-q)t} = \frac{\ln^{r+1} q}{q-1}\sum_{i=0}^{r-1}\binom{r+1}{i}\ln^{-i} q\frac{\ln^i t}{t} - \frac{(r+1)\ln q}{1-q}\frac{\ln^r t}{t}$$

Which can be rewritten as

$$\frac{\ln^r t}{t} = \frac{q-1}{(r+1)\ln q}D_q \ln^{r+1} t - \frac{\ln^r q}{r+1}\sum_{i=0}^{r-1}\binom{r+1}{i}\ln^{-i} q\frac{\ln^i t}{t}. \quad (3.14)$$

Multiplying both sides by E_q^{-tq} and then q-integrating from ϵ to x with taking the neutrix limit to be

$$\gamma_q^{(r)}(0,x) = \frac{q-1}{(r+1)\ln q}N-\lim_{\epsilon\to 0}\int_\epsilon^t D_q \ln^{r+1} tE_q^{-qt}d_qt - \frac{\ln^r q}{r+1}\sum_{i=0}^{r-1}\binom{r+1}{i}\ln^{-i} q\gamma_q^{(i)}(0,x)$$

Using the q-integration by parts rule

$$\int_a^b f(x)D_q g(x)d_qx = f(b)g(b) - f(a)g(a) - \int_a^b g(xq)D_q f(x)d_qx. \quad (3.15)$$

Would yield

$$\gamma_q^{(r)}(0,x) = \frac{q-1}{(r+1)\ln q}\left(\ln^{r+1} xE_q^{-xq} + \gamma_q^{(r+1)}(1,xq)\right) - \frac{\ln^r q}{r+1}\sum_{i=0}^{r-1}\binom{r+1}{i}\ln^{-i} q\gamma_q^{(i)}(0,x)$$

The equation (3.6) completes the proof of (3.12). To prove (3.13), put $x = 1/(1-q)$ in

(3.12).

Corollary 3.6. For $r = 1, 2, \cdots$, the rth derivative of the incomplete gamma function can be estimated by the equation

$$\gamma^{(r)}(0,x) = \frac{1}{r+1}\left(e^{-x}\ln^{r+1} x + \gamma^{(r+1)}(1,x)\right), \quad (3.16)$$

And for the gamma function by the equation

$$\Gamma^{(r)}(0) = \frac{\Gamma^{(r+1)}(1)}{r+1}. \quad (3.17)$$

Theorem 3.7. For $m = 1, 2, \cdots$, we have for incomplete q-gamma function

$$\gamma_q'(-m,x) = \frac{(-1)^m q^{\frac{m(m+1)}{2}}}{[m]_q!}\left(\frac{q-1}{2\ln q}(\ln^2 x E_q^{-x} + \gamma_q''(1,x)) + \ln q(E_1(x,q)+\gamma_q)\left[\frac{1}{2}+\frac{H_{m,q}}{1-q}\right]\right.$$
$$- \ln x E_q^{-x}\sum_{i=1}^{m}(-1)^i q^{-\frac{i(i-1)}{2}}[i-1]_q! x^{-i} - \frac{\ln q}{1-q}\sum_{i=1}^{m}\frac{H_{i,q}}{[i]_q}$$
$$\left.+\frac{\ln q}{1-q}E_q^{-x}\sum_{i=1}^{m}\frac{1}{[i]_q}\sum_{j=1}^{i}(-1)^j q^{-\frac{j(j+1)}{2}}[j-1]_q! x^{-j}\right), \tag{3.18}$$

And for q-gamma function

$$\Gamma_q'(-m) = \frac{(-1)^m q^{\frac{m(m+1)}{2}}}{[m]_q!}\left(\frac{q-1}{2\ln q}\Gamma_q''(1) + \gamma_q \ln q\left[\frac{1}{2}+\frac{H_{m,q}}{1-q}\right] - \frac{\ln q}{1-q}\sum_{i=1}^{m}\frac{H_{i,q}}{[i]_q}\right). \tag{3.19}$$

Proof. From the definition of q-derivative, we see that

$$\frac{\ln q}{q-1}t^{-m-1}E_q^{-tq} = t^{-m}E_q^{-tq}D_q \ln t$$

On q-integrating from $\epsilon \to x$ with taking the neutrix limit, we obtain

$$\frac{\ln q}{q-1}\gamma_q(-m,x) = N - \lim_{\epsilon\to 0}\int_\epsilon^x t^{-m}E_q^{-tq}D_q \ln t d_q t$$

On q-integrating by parts with using the q-derivative of two product functions

$D_q[f(t)g(t)] = g(t)D_q f(t) + f(tq)D_q g(t) = f(t)D_q g(t) + g(tq)D_q f(t)$ and the equation (3.6) would yield

$$\frac{\ln q}{q-1}\gamma_q(-m,x) = [m]_q\gamma_q'(-m,x) + q^m\gamma_q'(-m+1,x) + q^m x^{-m}\ln x E_q^{-x}$$

Which can be rewritten as

$$\gamma_q'(-m,x) = \frac{-q^m}{[m]_q}\gamma_q'(-m+1,x) - g(m)$$

Where

$$g(m) = \frac{\ln q}{1-q^m}\gamma_q(-m,x) + \frac{q^m}{[m]_q}x^{-m}\ln x E_q^{-x}$$

Inserting (3.3) into the above function yields

$$g(m) = \frac{\ln q}{1-q}\frac{(-1)^m q^{\frac{m(m+1)}{2}}}{[m]_q\cdot[m]_q!}\left(H_{m,q} - E_1(x,q) - \gamma_q - E_q^{-x}\sum_{i=1}^{m}(-1)^i q^{-\frac{i(i-1)}{2}}[i-1]_q! x^{-i}\right)$$
$$+ \frac{q^m}{[m]_q}x^{-m}\ln x E_q^{-x}$$

As similar as in Theorem 3.3, we can arrive at

$$\gamma_q'(-m,x) = \frac{(-1)^m q^{\frac{m(m+1)}{2}}}{[m]_q!}\gamma_q'(0,x) - \frac{(-1)^m q^{\frac{m(m+1)}{2}}}{[m]_q!}\sum_{i=1}^{m}(-1)^i q^{-\frac{i(i+1)}{2}}[i]_q! g(i)$$

Substituting $g(i)$ and (3.12) into the previous result would yield the desired result.

Corollary 3.8. For $m = 1, 2, \cdots$, we have for incomplete gamma function

$$\gamma'(-m,x) = \frac{(-1)^m}{m!}\left(\frac{1}{2}(\ln^2 xe^{-x} + \gamma''(1,x)) - H_m(E_1(x)+\gamma) + \sum_{i=1}^{m}\frac{H_i}{i} - \ln xe^{-x}\sum_{i=1}^{m}(-1)^i(i-1)!x^{-i} - e^{-x}\sum_{i=1}^{m}\frac{1}{i}\sum_{j=1}^{i}(-1)^j(j-1)!x^{-j}\right), \quad (3.20)$$

and for gamma function

$$\Gamma'(-m) = \frac{(-1)^m}{m!}\left(\frac{\Gamma''(1)}{2} - \gamma H_m + \sum_{i=1}^{m}\frac{H_i}{i}\right). \quad (3.21)$$

Theorem 3.9. For $m = 1, 2, \cdots$ and $r = 2, \cdots$, we have

$$\gamma_q^{(r)}(-m,x) = \frac{(-1)^m q^{\frac{m(m+1)}{2}}}{[m]_q!}\gamma_q^{(r)}(0,x) - \frac{(-1)^m q^{\frac{m(m+1)}{2}}}{[m]_q!}\sum_{i=1}^{m}(-1)^i q^{-\frac{i(i+1)}{2}}[i]_q!h(i,x), \quad (3.22)$$

and

$$\Gamma_q^{(r)}(-m) = \frac{(-1)^m q^{\frac{m(m+1)}{2}}}{[m]_q!}\Gamma_q^{(r)}(0) - \frac{(-1)^m q^{\frac{m(m+1)}{2}}}{[m]_q!}\sum_{i=1}^{m}(-1)^i q^{-\frac{i(i+1)}{2}}[i]_q!h(i,1/(1-q)) \quad (3.23)$$

,

Where

$$h(i,x) = \frac{r\ln q}{1-q^i}\gamma_q^{(r-1)}(-i,x) + \frac{q^i}{[i]_q}x^{-i}\ln^r xE_q^{-x} + \frac{\ln^r q}{1-q^i}\sum_{j=0}^{r-2}\binom{r}{j}\ln^{-j} q\gamma_q^{(j)}(-i,x). \quad (3.24)$$

Proof. The equation (3.14) can be read for $r = 2,3,\cdots$as

$$\frac{\ln^{r-1} t}{t} = \frac{q-1}{r\ln q}D_q\ln^r t - \frac{\ln^{r-1} q}{r}\sum_{i=0}^{r-2}\binom{r}{i}\ln^{-i} q\frac{\ln^i t}{t}.$$

Multiplying both sides by $t^{-m}E_q^{-tq}$ and then q-integrating from ϵ to x with taking the neutrix limit to be

$$\gamma_q^{(r-1)}(-m,x) = \frac{q-1}{r\ln q}N\text{-}\lim_{\epsilon\to 0}\int_\epsilon^x D_q(\ln^r t)t^{-m}E_q^{-tq}d_qt - \frac{\ln^{r-1} q}{r}\sum_{i=0}^{r-2}\binom{r}{i}\ln^{-i} q\gamma_q^{(i)}(-m,x)$$

On q-integrating by parts with using the q-derivative of two product functions and the equation (3.6) would yield

$$\gamma_q^{(r-1)}(-m,x) = \frac{q-1}{r\ln q}\left(q^m x^{-m}\ln^r xE_q^{-x} + q^m\gamma_q^{(r)}(-m+1,x) + [m]_q\gamma_q^{(r)}(-m,x)\right) - \frac{\ln^{r-1} q}{r}\sum_{i=0}^{r-2}\binom{r+1}{i}\ln^{-i} q\gamma_q^{(i)}(-m,x)$$

Which can be read as

$$\gamma_q^{(r)}(-m,x) = \frac{-q^m}{[m]_q}\gamma_q^{(r)}(-m+1,x) - h(m,x)$$

Where $h(m,x)$ is defined as in (3.24). As similar as in Theorem 3.3, we can arrive at the desired results.

Corollary 3.10. For $m,r = 1,2,\cdots$, we have

$$\gamma^{(r)}(-m,x) = \frac{(-1)^m}{(r+1)m!}\left(e^{-x}\ln^{r+1}x + \gamma^{(r+1)}(1,x)\right) + \frac{(-1)^m}{m!}\sum_{i=1}^{m}(-1)^i(i-1)!(r\gamma^{(r-1)}(-i,x) + x^{-i}\ln^r xe^{-x}), \qquad (3.25)$$

And

$$\Gamma^{(r)}(-m) = \frac{(-1)^m\Gamma^{(r+1)}(1)}{(r+1)m!} + \frac{r(-1)^m}{m!}\sum_{i=1}^{m}(-1)^i(i-1)!\Gamma^{(r-1)}(-i \qquad .(3.26)$$

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